Weighted balanced realization and model reduction for nonlinear systems

Daisuke Tsubakino and Kenji Fujimoto

Abstract— **In this paper a weighted balanced realization and model reduction for nonlinear systems is proposed which is based on singular value analysis of nonlinear Hankel operators. In the proposed model reduction procedure, an important property, stability is preserved. A numerical example is also shown. This result is expected to be a basis for weighted model order reduction for nonlinear systems.**

Keywords— **nonlinear systems, frequency weighted model reduction, weighted balanced realization**

I. Introduction

A nonlinear extension of the state-space concept of balanced realizations was proposed in [1] by introducing controllability and observability functions which are nonlinear counterparts of controllability and observability Gramians. Since then, many results on global balancing [2], modifications [3], computational issues [4], [5] and related minimality considerations [6], [7] have appeared in the literature. The authors also worked on both input-output and state-space characterizations of balanced realization based on singular value analysis analysis of nonlinear Hankel operators, e.g. [8], [9]. Although it still needs a lot of (computational) effort to calculate a nonlinear reduced order model in a practical situation, basic algorithms for model order reduction based on balanced realization were already established.

On the other hand, in the linear case, there is a technique called frequency weighted model order reduction which preserves the input behavior with respect to a certain frequency bandwidth. This technique is utilized quite often in the real world application. For instance, the behavior of a model with respect to too high frequencies does not coincide with that of the actual plant, so the input-output behavior with respect to lower frequencies should be preserved. A basic framework of frequency weighted balanced truncation in the linear case was proposed in [10]. However, in this procedure, some important properties, such as controllability, observability, and stability are not preserved even when the original system possesses these properties. Sreeram et al. [11] proposes a modified approach which guarantees stability of the reduced order model. The purpose of the present paper is to establish a nonlinear balanced truncation method by extending these linear case results to the nonlinear case. In particular we will derive a weighted nonlinear model order reduction method based on our former work [9] preserving stability of the original system as the Sreeram's result.

D. Tsubakino is with the Department of Aeronautics and Astronautics, Graduate School of Engineering, University of Tokyo, Hongo, Bunkyo-ku, Toyko 133-8656, Japan tsubakino@flab.eng.isas.jaxa.jp

K. Fujimoto is with the Department of Mechanical Science and Engineering, Graduate School of Engineering Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan k.fujimoto@ieee.org

II. Preliminaries

In this section we refer to nonlinear balanced truncation [9] and linear frequency weighted balanced truncation [11], respectively.

A. Balanced realization and model reduction for nonlinear systems

In this section, we refer to some preliminary results on nonlinear balanced realization. Consider an input-affine, time invariant, asymptotically stable nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^r$. Its controllability function $L_c(x)$ and observability function $L_o(x)$ are defined by

$$L_c(\xi) := \inf_{\substack{u \in L_2^- \\ x(-\infty)=0, x(0)=\xi}} \frac{1}{2}\|u\|^2_{L_2^m}$$
$$L_o(\xi) := \frac{1}{2}\|y\|^2_{L_2^r},\ x(0) = \xi,\ u = 0.$$

In the linear case,

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}P^{-1}x,\ L_o(x) = \frac{1}{2}x^{\mathrm{T}}Qx$$

hold with the controllability and observability Gramians P and Q. The functions $L_c(x)$ and $L_o(x)$ fulfill the following Hamilton-Jacobi equations.

$$\frac{\partial L_c(x)}{\partial x}f(x) + \frac{1}{2}\frac{\partial L_c(x)}{g}(x)g(x)^{\mathrm{T}}\frac{\partial L_c}{\partial x}^{\mathrm{T}} = 0$$
$$\frac{\partial L_o(x)}{\partial x}f(x) + \frac{1}{2}h(x)^{\mathrm{T}}h(x) = 0$$

Here $\dot{x} = -f - gg^{\mathrm{T}}(\partial L_c(x)/\partial x)^{\mathrm{T}}$ is asymptotically stable in a neighborhood W of the origin.

Theorem 1: [1] Assume that $\dot{x} = f(x)$ is asymptotically stable in a neighborhood W of the origin. If the system is *zero-state observable and L_o exists and is smooth on W, then $L_o(x) > 0,\ \forall x \in W, x \neq 0$*

Next, we review nonlinear balanced realization.

Theorem 2: [9] Suppose that $L_c(x)$ and $L_o(x)$ exist and that Hankel singular values of the Jacobian linearization of Σ are nonzero and distinct. Then there exist a neighborhood U of the origin and a coordinate transformation $x = \Phi(z)$ on U converting the system into the following form

$$L_c(\Phi(z)) = \frac{1}{2}\sum_{i=1}^{n}\frac{z_i^2}{\sigma_i(z_i)}$$
$$L_o(\Phi(z)) = \frac{1}{2}\sum_{i=1}^{n} z_i^2\sigma_i(z_i).$$

Here $\sigma_i(z_i)$'s are Hankel singular value functions of Σ. This realization is called nonlinear balanced realization which has the following properties.

$$z_i = 0 \Leftrightarrow \frac{\partial L_c(\Phi(z))}{\partial z_i} = \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0$$
$$\sigma_i^2(z_i) = \frac{L_o(\Phi(0,\dots,0,z_i,0,\dots,0))}{L_c(\Phi(0,\dots,0,z_i,0,\dots,0))}$$

The Hankel singular value functions $\sigma_i(z_i)$'s represent the importance of the state variables z_i's with respect to the input-output behavior of the system. Therefore we can get a reduced order model by truncating important states. This technique is called balanced truncation. Let Σ^a denote a reduced order model with state variables $z^a = (z_1,\dots,z_k)$.

Theorem 3: [9] The controllability and observability functions $L_c^a(z^a)$ and $L_o^a(z^a)$ of Σ^a satisfy

$$L_c^a(z^a) = L_c(\Phi(z^a,0))$$
$$L_o^a(z^a) = L_o(\Phi(z^a,0)).$$

Furthermore, Hankel singular values $\sigma_i^a(z_i^a)$ of Σ^a satisfy

$$\sigma_i^a(z_i^a) = \sigma_i(z_i), \quad i = 1,\dots,k.$$

These theorems imply that some important properties of Σ are preserved. Local asymptotic stability is also preserved.

B. Frequency weighted balanced realization and model reduction of linear systems

In the linear systems theory, frequency weighted balanced realization was introduced by Enns[10] and it was modified by Sreeram *et al.*[11] to guarantee stability of the reduced order model. Consider a following asymptotically stable and minimal linear system

$$\Sigma : \begin{cases}\dot{x} = Ax + Bu\\ y = Cx\end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Let the state-space realization of the input and output weights be given by

$$W_i : \begin{cases}\dot{x}_i = A_i x_i + B_i u_i\\ y_i = C_i x_i + D_i u_i\end{cases},\ W_o : \begin{cases}\dot{x}_o = A_o x_o + B_o u_o\\ y_o = C_o x_o + D_o u_o\end{cases} \tag{3}$$

where these realizations are minimal and asymptotically stable. Therefore, the state-space realization of the augmented system $W_o\Sigma W_i$ is given by

$$W_i\Sigma W_o : \begin{cases}\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u\\ y = \bar{C}\bar{x}\end{cases} \tag{4}$$

$$\bar{A} = \begin{pmatrix} A & 0 & BC_i\\ B_oC & A_o & 0\\ 0 & 0 & A_i\end{pmatrix}, \bar{B} = \begin{pmatrix} BD_i\\ 0\\ B_i\end{pmatrix}$$
$$\bar{C} = \begin{pmatrix} D_oC & 0 & C_o\end{pmatrix},\ \bar{x}^{\mathrm{T}} = \begin{pmatrix} x^{\mathrm{T}} & x_o^{\mathrm{T}} & x_i^{\mathrm{T}}\end{pmatrix}.$$

We assume that there are no pole-zero cancellation in $W_o\Sigma W_i$. The controllability and observability Gramians of the augmented system $\bar{P}$ and $\bar{Q}$ fulfill Lyapunov equations.

$$\bar{A}\bar{P} + \bar{P}\bar{A}^{\mathrm{T}} + \bar{B}\bar{B}^{\mathrm{T}} = 0 \tag{5}$$
$$\bar{Q}\bar{A} + \bar{A}^{\mathrm{T}}\bar{Q} + \bar{C}^{\mathrm{T}}\bar{C} = 0 \tag{6}$$

In Sreeram's technique, the weighted controllability and observability Gramians P and Q corresponding to Σ are defined by

$$P := P_{11} - P_{13}P_{33}^{-1}P_{13}^{\mathrm{T}},\ Q := Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\mathrm{T}} \tag{7}$$

where P_{ij}'s and Q_{ij}'s satisfy

$$\bar{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13}\\ P_{12}^{\mathrm{T}} & P_{22} & P_{23}\\ P_{13}^{\mathrm{T}} & P_{23}^{\mathrm{T}} & P_{33}\end{pmatrix},\ \bar{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13}\\ Q_{12}^{\mathrm{T}} & Q_{22} & Q_{23}\\ Q_{13}^{\mathrm{T}} & Q_{23}^{\mathrm{T}} & Q_{33}\end{pmatrix}. \tag{8}$$

Since there are no pole-zero cancellation, P and Q are positive definite. Then we can calculate the balancing coordinate transformation $x = Tz$ for the weighted balanced realization satisfying

$$T^{-1}PT^{-\mathrm{T}} = T^{\mathrm{T}}QT = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n). \tag{9}$$

Let us verify the stability of the reduced order model. Equations (5)–(7), we can obtain the following Lyapunov equations

$$AP + PA^{\mathrm{T}} + XX^{\mathrm{T}} = 0 \tag{10}$$
$$QA + A^{\mathrm{T}}Q + Y^{\mathrm{T}}Y = 0 \tag{11}$$

where $X := BD_i - P_{13}P_{33}^{-1}B_i,\ Y := D_oC - C_oQ_{22}^{-1}Q_{12}$. Since P and Q are positive definite and A is stable, (A, X) is controllable and (Y, A) is observable. Here, let (A^r, X^r, Y^r) denote the reduced order model truncated from the system (A, X, Y). Since the truncation procedure applied here is a conventional (non-weighted) balanced truncation, the asymptotic stability of the original system is preserved, that is, A^r is stable.

When $W_o = \mathrm{id}$ holds[1], i.e., only the input weight is applied, the weighted observability Gramian Q defined in Equation (8) coincides with the unweighted observability Gramian of the original system and $Y = C$ holds. Similarly, when $W_i = \mathrm{id}$, the weighted controllability Gramian P coincides with the unweighted one and $X = B$.

III. Main Results

This section generalizes nonlinear balanced realization and nonlinear balanced truncation technique to include input and output weights. Stability of the reduced order model will be proved.

A. Nonlinear weighted balanced realization

Let us extend the technique for linear systems to nonlinear systems here. Consider a time invariant, input-affine, nonlinear system,

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{12}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$. Σ is assumed to be asymptotically stable in a neighborhood of 0. Furthermore, we employ time invariant, input-affine, asymptotically stable minimal nonlinear systems W_i and W_o for the input and output weighting operators

$$\begin{aligned} W_i &: \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\ y_i = h_i(x_i) + d_i(x_i)u_i \end{cases}, \\ W_o &: \begin{cases} \dot{x}_o = f_o(x_o) + g_o(x_o)u_o \\ y_o = h_o(x_o) + d_o(x_o)u_o \end{cases}. \end{aligned} \tag{13}$$

Then the state-space realization for the augmented system $W_o \circ \Sigma \circ W_i$ is given by

$$W_o \circ \Sigma \circ W_i : \begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}(\bar{x})u \\ y = \bar{h}(\bar{x}) \end{cases} \tag{14}$$

$$\begin{aligned} \bar{f}(\bar{x}) &= \begin{pmatrix} f(x) + g(x)h_i(x_i) \\ f_o(x_o) + g_o(x_o)h(x) \\ f_i(x_i) \end{pmatrix}, \\ \bar{g}(\bar{x}) &= \bar{g}(x, x_i) = \begin{pmatrix} g(x)d_i(x_i) \\ 0 \\ g_i(x_i) \end{pmatrix}, \\ \bar{h}(\bar{x}) &= \bar{h}(x, x_o) = h_o(x_o) + d_o(x_o)h(x), \\ \bar{x}^{\mathrm{T}} &= \begin{pmatrix} x^{\mathrm{T}} & x_o^{\mathrm{T}} & x_i^{\mathrm{T}} \end{pmatrix}. \end{aligned}$$

Now, assume that the augmented system is asymptotically stable and minimal. The controllability function $\bar{L}_c(\bar{x})$ and observability function $\bar{L}_o(\bar{x})$ of the augmented system fulfill the following Hamilton-Jacobi equations.

$$\frac{\partial \bar{L}_c(\bar{x})}{\partial \bar{x}} \bar{f}(\bar{x}) + \frac{1}{2} \frac{\partial \bar{L}_c(\bar{x})}{\partial \bar{x}} \bar{g}(\bar{x}) \bar{g}(\bar{x})^{\mathrm{T}} \frac{\partial \bar{L}_c(\bar{x})}{\partial \bar{x}}^{\mathrm{T}} = 0 \tag{15}$$

$$\frac{\partial \bar{L}_o(\bar{x})}{\partial \bar{x}} \bar{f}(\bar{x}) + \frac{1}{2} \bar{h}(\bar{x})^{\mathrm{T}} \bar{h}(\bar{x}) = 0 \tag{16}$$

[1]The symbol id denotes the identity operator.

Our purpose is to obtain the weighted controllability function $L_c(x)$ and the observability function $L_o(x)$ corresponding to Σ truncated from $\bar{L}_c(\bar{x})$ and $\bar{L}_o(\bar{x})$, respectively, as in the linear case. We use state constraints on x_i and x_o to define these functions, i.e.,

$$L_c(x) := \bar{L}_c(x, \tilde{x}_{oc}(x), 0), \quad L_o(x) := \bar{L}_o(x, \tilde{x}_{oo}(x), 0) \tag{17}$$

where $\tilde{x}_{oc}(x)$ and $\tilde{x}_{oo}(x)$ are constraints on the state x_o satisfying

$$\left. \frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x_o} \right|_{\substack{x_o = \tilde{x}_{oc}(x) \\ x_i = 0}} \equiv 0, \tag{18}$$

$$\left. \frac{\partial \bar{L}_o(x, x_o, x_i)}{\partial x_o} \right|_{\substack{x_o = \tilde{x}_{oo}(x) \\ x_i = 0}} \equiv 0. \tag{19}$$

According to the implicit function theorem, it is guaranteed that there exist $\tilde{x}_{oc}(x)$ and $\tilde{x}_{oo}(x)$ in a neighborhood of a point where the Hessians of $\bar{L}_c(x, x_o, 0)$ and $\bar{L}_o(x, x_o, o)$ with respect to x_o are positive definite, i.e.,

$$\left. \frac{\partial^2 \bar{L}_c(\bar{x})}{\partial x_o^2} \right|_{x_i = 0} > 0, \quad \left. \frac{\partial^2 \bar{L}_o(\bar{x})}{\partial x_o^2} \right|_{x_i = 0} > 0. \tag{20}$$

Since the Gramians of the Jacobian linearization of the augmented system are positive definite, the functions $\tilde{x}_{oc}(x)$ and $\tilde{x}_{oo}(x)$ exist in a neighborhood of the origin. For convenience, we write $\tilde{x}_{oc}(x)$ and $\tilde{x}_{oo}(x)$ as $\tilde{x}_{oc}$ and $\tilde{x}_{oo}$. In the linear case, using Schur compliment, we can verify

$$\begin{aligned} \tilde{x}_{oc} &= \frac{1}{2}(P_{12} - P_{13}P_{33}^{-1}P_{23}^{\mathrm{T}})^{\mathrm{T}}(P_{11} - P_{13}P_{33}^{-1}P_{13}^{\mathrm{T}})^{-1}x, \\ \tilde{x}_{oo} &= -\frac{1}{2}Q_{22}^{-1}Q_{12}^{\mathrm{T}}x. \end{aligned} \tag{21}$$

They lead to

$$\begin{aligned} L_c(x) &= \frac{1}{2}x^{\mathrm{T}}(P_{11} - P_{13}P_{33}^{-1}P_{13}^{\mathrm{T}})^{-1}x, \\ L_o(x) &= \frac{1}{2}x^{\mathrm{T}}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\mathrm{T}})\, x \end{aligned} \tag{22}$$

which reveals that this definition of the constraints $\tilde{x}_{oc}(x)$ and $\tilde{x}_{oo}(x)$ is a natural nonlinear generalization of the linear case result.

Application of unweighted balanced realization procedure with respect to the weighted controllability and observability functions $L_c(x)$ and $L_o(x)$ yields weighted balanced realization. That is, if the product $(P_{11} - P_{13}P_{33}^{-1}P_{13}^{\mathrm{T}})(Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\mathrm{T}})$, which are calculated from the Gramians of the Jacobian linearization of the augmented model, has nonzero and distinct eigenvalues, then Theorem 2 implies that there exist a neighborhood U of the origin and a coordinate transformation $x = \Phi(z)$ on U converting the weighted controllability and observability functions into the following form

$$L_c(\Phi(z)) = \frac{1}{2}\sum_{i=1}^{n}\frac{z_i^2}{\sigma_i(z_i)}, \tag{23}$$

$$L_o(\Phi(z)) = \frac{1}{2}\sum_{i=1}^{n} z_i^2\sigma_i(z_i). \tag{24}$$

Then we can get a weighted reduced order model of Σ by truncation. In the linear case, these equations reduce to (9). See [8], [9] for the detail. Thus, this procedure seems to be natural extension of linear weighted balanced realization.

Remark 1: As in the linear case, when $W_o = \mathrm{id}$, that is, only the input weight is applied, the weighted observability function $L_o(x)$ coincides with the unweighted observability function of the original system Σ. Similarly, when $W_i = \mathrm{id}$, the weighted controllability function $L_c(x)$ coincides with the unweighted one.

B. Model reduction and stability

Suppose that we already have a coordinate transformation $x = \Phi(z)$ for weighted balanced realization and that

$$\min \sigma_k(z_k) \gg \max \sigma_k(z_k), \quad \forall z \in W$$

holds with a neighborhood W of the origin. Then the state variables $(z_1, \dots, z_k)$ play more important roles than $(z_{k+1}, \dots, z_n)$. According to this partition, divide the coordinates into two parts as

$$\begin{aligned} z^a &:= (z_1, \dots, z_k) \in \mathbb{R}^k \\ z^b &:= (z_{k+1}, \dots, z_n) \in \mathbb{R}^{n-k} \\ \begin{pmatrix} f^{za}(z) \\ f^{zb}(z) \end{pmatrix} &:= f^z(z) = \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} f(\Phi(z)) \\ \begin{pmatrix} g^{za}(z) \\ g^{zb}(z) \end{pmatrix} &:= g^z(z) = \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} g(\Phi(z)). \end{aligned}$$

Then we obtain a weighted reduced order model Σ^a of Σ on W as

$$\Sigma^a : \begin{cases} \dot{z}^a = f^{za}(z^a, 0) + g^{za}(z^a, 0)u \\ y = h(\Phi(z^a, 0)) \end{cases}. \tag{25}$$

Next, we show the stability of the weighted reduced model Σ^a. In order to prove the stability, we provide the following lemma which is a slightly modified version of the Morse's lemma [12],

Lemma 1: Let α be a smooth vector valued function : $\xi \mapsto \alpha(\xi) \in \mathbb{R}^{1\times p}$, $\alpha(0) = 0$ in a neighborhood V of 0. Then there exist a smooth matrix $\Gamma \in \mathbb{R}^{n\times p}$ defined on V such that

$$\alpha(\xi) = \xi^{\mathrm{T}}\Gamma(\xi). \tag{26}$$

Using this lemma, our main result can be obtained. The following theorem guarantees local asymptotic stability of the reduced order model in (frequency) weighted balanced truncation.

Theorem 4: Suppose that the Jacobian linearization of the augmented model is controllable and that the original system Σ is asymptotically stable. Then there exists a neighborhood U of origin such that the weighted reduced model Σ^a is asymptotically stable on U.

Proof: Substituting the weighted controllability and observability functions for Equations (15) and (16), we can obtain the following equations

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2}\left\| \left.\frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial (x, x_i)}\right|_{\substack{x_o=\tilde{x}_{oc} \\ x_i=0}} \begin{pmatrix} g(x)D_i \\ B_i \end{pmatrix} \right\|^2 = 0, \tag{27}$$

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2}\tilde{h}(x)^{\mathrm{T}}\tilde{h}(x) = 0 \tag{28}$$

where $B_i := g_i(0)$, $D_i := d_i(0)$, $\tilde{h}(x) := h_o(\tilde{x}_{oo}) + d_o(\tilde{x}_{oo})h(x)$. Here we use the following relation

$$\begin{aligned} \frac{\partial L_c(x)}{\partial x} &= \frac{\partial}{\partial x}(\bar{L}_c(x, \tilde{x}_{oc}, 0)) \\ &= \left.\left(\frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x} + \frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x_o}\frac{\partial \tilde{x}_{oc}}{\partial x}\right)\right|_{\substack{x_o=\tilde{x}_{oc} \\ x_i=0}} \\ &= \left.\frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x}\right|_{\substack{x_o=\tilde{x}_{oc} \\ x_i=0}} \end{aligned}$$

and a similar equation where $L_c(x)$ and $\tilde{x}_{oc}$ are replaced by $L_o(x)$ and $\tilde{x}_{oo}$. Now, we define a coordinate transformation $\xi = \Psi(x)$ as follows.

$$\xi = \Psi(x) := \left.\frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x}^{\mathrm{T}}\right|_{\substack{x_o=\tilde{x}_{oc} \\ x_i=0}} = \frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} \tag{29}$$

This transformation fulfills $\Psi(0) = 0$. Let $x_i \in \mathbb{R}^{n_i}$ and define $\alpha(\xi) \in \mathbb{R}^{1\times n_i}$ as

$$\alpha(\xi) := \frac{\partial \bar{L}_c}{\partial x_i}(\Psi^{-1}(\xi), \tilde{x}_{oc}(\Psi^{-1}(\xi)), 0). \tag{30}$$

Since the Hessian of $\bar{L}_c(\bar{x})$ with respect to x is positive definite at the origin, the implicit function theorem guarantees that there exist a neighborhood V of the origin and that $\xi = 0 \Rightarrow \Psi^{-1}(0) = 0$ on V. Then, we can apply Lemma 1 to $\alpha(\xi)$ on V because $\alpha(0) = 0$. There exists a matrix $\bar{\Gamma}(\xi) \in \mathbb{R}^{n\times n_i}$ such that

$$\alpha(\xi) = \xi^{\mathrm{T}}\bar{\Gamma}(\xi).$$

Let $\Gamma(x) := \bar{\Gamma}(\Psi(x))$, then we obtain following equation.

$$\left.\frac{\partial \bar{L}_c(x, x_o, x_i)}{\partial x_i}\right|_{\substack{x_o=\tilde{x}_{oc} \\ x_i=0}} = \frac{\partial L_c(x)}{\partial x}\Gamma(x). \tag{31}$$

Finally, substituting Equation (31) for Equation (27), we obtain the following Hamilton-Jacobi equation

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{\partial L_c(x)}{\partial x}\tilde{g}(x)\tilde{g}(x)^{\mathrm{T}}\frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} = 0 \tag{32}$$

where $\tilde{g}$ is defined by

$$\tilde{g}(x) := \begin{pmatrix} I_n & \Gamma(x) \end{pmatrix} \begin{pmatrix} g(x)D_i \\ B_i \end{pmatrix}. \tag{33}$$

Therefore, let system Σ' be given by

$$\Sigma' \begin{cases} \dot{x} = f(x) + \tilde{g}(x)u \\ y = \tilde{h}(x) \end{cases}, \tag{34}$$

then, from Equations (28) and (32), the weighted balanced truncation problem of Σ corresponds to the unweighted balanced truncation problem of Σ' in a neighborhood of the origin as in the linear case. Since unweighted balanced truncation guarantees the stability of the reduced order model, the reduced model Σ^a is also asymptotically stable. This completes the proof. ■

Furthermore, we can show following theorem on global stability of the reduced order model.

Theorem 5: Suppose that $\bar{L}_o(\bar{x})$ is radially unbounded and positive definite, the coordinate transformation $x = \Phi(z)$ for weighted balanced realization is defined globally and $\tilde{x}_{oo}(x)$ is defined globally. Then the reduced order model Σ^a is globally stable.

Proof: The Hamilton-Jacobi equation (28) with respect to $L_o(x)$ reduces to

$$\frac{\partial L_o(\Phi(z))}{\partial x} f(\Phi(z)) + \frac{1}{2} h(\Phi(z))^{\mathrm{T}} h(\Phi(z)) = 0$$

$$\frac{\partial L_o(\Phi(z))}{\partial x} \frac{\partial \Phi(z)}{\partial z} \frac{\partial \Phi(z)}{\partial z}^{-1} f(\Phi(z)) + \frac{1}{2} h(\Phi(z))^{\mathrm{T}} h(\Phi(z)) = 0$$

$$\therefore \quad \frac{\partial L_o(\Phi(z))}{\partial z} f^z(z) + \frac{1}{2} h(\Phi(z))^{\mathrm{T}} h(\Phi(z)) = 0. \tag{35}$$

Substituting $z = (z^a, 0)$, we obtain

$$\frac{\partial L_o(\Phi(z^a,0))}{\partial z} f^{za}(z^a) + \frac{1}{2} h(\Phi(z^a,0))^{\mathrm{T}} h(\Phi(z^a,0)) = 0. \tag{36}$$

Let $L_o^a(z^a) := L_o(\Phi(z^a,0))$, its time-derivative along to a solution z^a is

$$\begin{aligned} \frac{\mathrm{d}L_o^a(z^a)}{\mathrm{d}t} &= \frac{\partial L_o(\Phi(z^a,0))}{\partial z} f^{za}(z^a) \\ &= -\frac{1}{2} \|h(\Phi(z^a,0))\|_{\mathbb{R}^n}. \end{aligned} \tag{37}$$

The right hand side of this equation is negative semi-definite and this implies that $L_o^a(z^a)$ is a Lyapunov function of Σ^a defined globally. Therefore, Σ^a is globally stable. ■

Remark 2: The assumption that $\bar{L}_o(\bar{x})$ is positive definite is automatically satisfied when the augmented system $W_o \circ \Sigma \circ W_i$ is asymptotically stable and zero-state observable.

Corollary 1: In Theorem 5, if the reduced order model Σ^a is zero-state observable, then Σ^a is global asymptotically stable.

Proof: The zero-state observability of Σ^a implies that

$$y = h(\Phi(z^a,0)) \equiv 0 \Rightarrow z^a \equiv 0. \tag{38}$$

Thus, the time-derivative of $L_o^a(z^a)$ is negative definite for Equation (37). Since $L^a(z^a)$ is assumed to be positive definite, Lyapunov theory guarantees globally asymptotical stability of Σ^a. ■

IV. Numerical simulation

In this section, we apply the proposed procedure to a double pendulum depicted in Figure 1. In this study, we

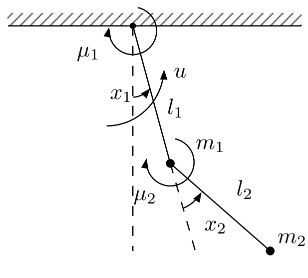

Fig. 1. The double pendulum

take the physical parameters as follows: the masses located at the end of the first and second links: $m_1 = 5.0, m_2 = 2.0$, the lengths of the first and second links: $l_1 = 5.0, l_2 = 1.0$, the friction coefficients of the first and second links: $\mu_1 = 25, \mu_2 = 0.05$ and the gravity constant: $g = 9.8$. The input u is the torque applied to the joint of the first link and the output y denotes the horizontal diaplacement of the mass 2:

$$y = l_1 \sin x_1 + l_2 \sin(x_1 + x_2). \tag{39}$$

The state-space realization of this system can be described as the form (1) with a 4 dimensional state $x = (x_1, x_2, x_3, x_4) := (x_1, x_2, \dot{x}_1, \dot{x}_2)$. The system has two natural freuencies as one can observe from the response of the output signal. Here we consider a problem to obtain a reduced order model approximating the lower natural frequency output behavior of the original model. Therefore, the following low pass filter is employed as an output weighting operator.

$$W_o : \begin{cases} \dot{x}_o = \begin{pmatrix} 0 & 1 \\ -16.9 & -2.6 \end{pmatrix} x_o + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_o \\ y_o = \begin{pmatrix} 1.69 & 0 \end{pmatrix} x_o \end{cases} \tag{40}$$

No input weight is used, i.e. $W_i = \mathrm{id}$.

Figures 2 and 3 plot the Hankel singular value functions obtained by the normal (internal) balanced and the weighted balanced realization respectively. Though the system is nonlinear, the singular value functions are almost constant in both cases. As noted in the figure, the horizontal axis denotes a parameter whose absolute value represents distance from the origin s, the line with '△' denotse $\sigma_1(s)$, the line with '▽' denotes $\sigma_2(s)$, the line

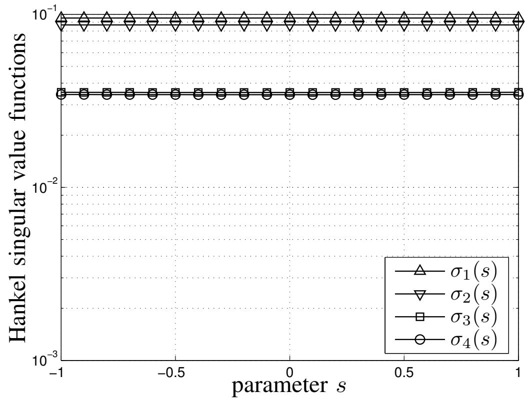

Fig. 2. Normal Hankel singular value functions

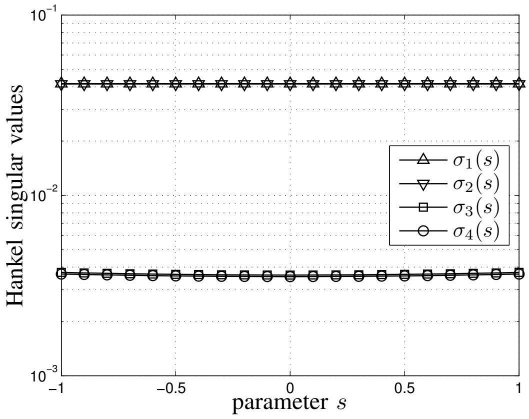

Fig. 3. Weighted Hankel singular value functions

with '□' denotes $\sigma_3(s)$ and the line with '◯' denotes $\sigma_4(s)$. In Figure 2, the singular value functions are close to each other. This fact implies that all state variables are closely related with the input and output behavior. Therefore, we can not obtain a reduced order model by truncation. On the other hand, we can obsearve that $\sigma_2(s) \gg \sigma_3(s)$ holds by introducing the weight in Figure 3. This fact implies that $\sigma_1(s)$ and $\sigma_2(s)$ are strongly related with *weighted (low frequency)* output compared with $\sigma_3(s)$ and $\sigma_4(s)$. Therefore we can obtain the reduced order model by truncating the states correspondings to $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ and the dimension of reduced order model is 2.

Finally, we compare output responses of the orignal and weighted reduced order models respect to impulsive input. Figure 4 depicts the output responses. The black line represents the ouput of the original model and the gray one represents that of the weighted reduced order model. It can be readily observed that the gray line approximates the low frequency movement of the black one. It is sure that the gray line can capture the low frequency movement of the black line. This result states the effect of the proposed

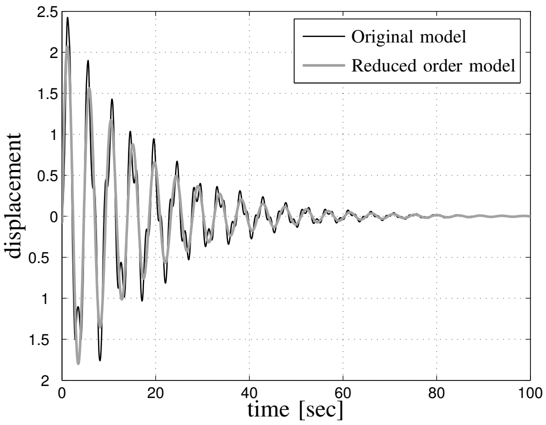

Fig. 4. Responses of the horizontal displacement

procedure.

V. Conclusion

In this paper, (frequency) weighted balanced realization and the corresponding model order reduction technique are proposed. This is a natural nonlinear generalization of the existing linear case results. It is proved that the reduced order system becomes stable when the original system is so. Furthermore, the numerical simulation has demonstrated its effect.

References

[1] J. M. A. Scherpen, "Balancing for nonlinear systems," *Systems & Control Letters*, vol. 21, pp. 143–153, 1993.

[2] E. I. Verriest and W. S. Gray, "Discrete-time nonlinear balancing," in *Proc. 5th IFAC Symp. Nonlinear Control Systems*, 2001, pp. 515–520.

[3] J. Hahn and T. F. Edgar, "Reduction of nonlinear models using balancing of empirical gramians and Galerkin projections," in *Proc. American Control Conference*, 2000, pp. 2864–2868.

[4] A. J. Newman and P. S. Krishnaprasad, "Computation for nonlinear balancing," in *Proc. 37th IEEE Conf. on Decision and Control*, 1998, pp. 4103–4104.

[5] K. Fujimoto and T. Tsubakino, "On computation of nonlinear balanced realization and model reduction," 2005, submitted.

[6] J. M. A. Scherpen and W. S. Gray, "Minimality and local state decompositions of a nonlinear state space realization using energy functions," *IEEE Trans. Autom. Contr.*, vol. AC-45, no. 11, pp. 2079–2086, 2000.

[7] K. Perev, "Reachability and reachability grammians for a class of nonlinear systems," 2000, preprint, Submitted.

[8] K. Fujimoto and J. M. A. Scherpen, "Nonlinear input-normal realizations based on the differential eigenstructure of Hankel operators," *IEEE Trans. Autom. Contr.*, vol. 50, no. 1, pp. 2–18, 2005.

[9] ——, "Nonlinear balanced realization based on singular value analysis of Hankel operators," in *Proc. 42nd IEEE Conf. on Decision and Control*, 2003, pp. 6072–6077.

[10] D. F. Enns, "Model reduction with balanced realizations: an error bound and frequency weighted generalization," in *Proc. 23rd IEEE Conf. on Decision and Control*, 1984, pp. 127–132.

[11] V. Sreeram, B. D. O. Anderson, and A. G. Madievski, "New results on frequency weighted balanced reduction technique," in *Proc. American Control Conference*, 1995, pp. 4004–4009.

[12] J. W. Milnor, *Morse Theory*. New Jersey: Annals of Math. Stud. 51, Princeton University Press, 1963.